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NOTE

THE HELLY-TYPE PROPERTY OF NON-TRIVIAL INTERVALS ON A TREE

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The purpose of this note is to prove a counterpart of the Helly property for a family of non-trivial intervals on a tree.

Let S be a finite set and $\mathcal{R} = (R_i \mid i \in I)$ be a family of subsets of S. $\mathcal{R}$ is said to satisfy the *Helly property* if $R_i \cap R_j \neq \emptyset$ for all $i, j \in I$ implies that

$$\bigcap_{j \in I} R_j \neq \emptyset.$$

A family of closed intervals on the line and on a tree satisfy the Helly property [1].

Let us assume that all intervals are *non-trivial* that is they are different from points. In this case, a family $\mathcal{R}$ of non-trivial intervals is said to satisfy the *non-trivial Helly property* if $R_i \cap R_j$ is a non-trivial interval for all $i, j \in I$ implies that

$$\bigcap_{j \in I} R_j \text{ is a non-trivial interval.}$$

A finite family $\mathcal{R}$ of non-trivial intervals on the line satisfies the non-trivial Helly property and the family $\mathcal{R} = (R_i \mid i = 0, 1, 2, \ldots)$, where $R_0 = [0, 1]$ and $R_i = [1 - 1/n, 2]$ $(i = 1, 2, \ldots)$ shows that the property may not be true for an infinite family of non-trivial intervals.

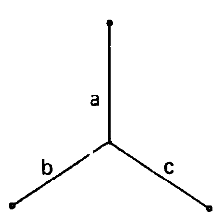

Fig. 1.

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Let $\mathcal{R}$ consist of the following intervals $R_1=(a,c)$, $R_2=(c,b)$, $R_3=(b,a)$ on the tree shown in Fig. 1. Every two intervals in $\mathcal{R}$ have a non-trivial intersection but there exists no common non-trivial interval for all intervals in $\mathcal{R}$.

We now formulate and prove the main result of this note.

Observation. *Let $\mathcal{R}=(R_i \mid i\in I)$ be a finite family of non-trivial intervals on a tree. If every three intervals R_i, R_j, R_k $(i,j,k\in I)$ have a non-trivial intersection, then $\bigcap_{i\in I} R_i$ is a non-trivial interval.*

Proof. We prove the theorem by induction on the number of intervals. Assume that $\bigcap_{i\in K} R_i = Q$, where $|K|=k<|I|$ is a non-trivial interval.

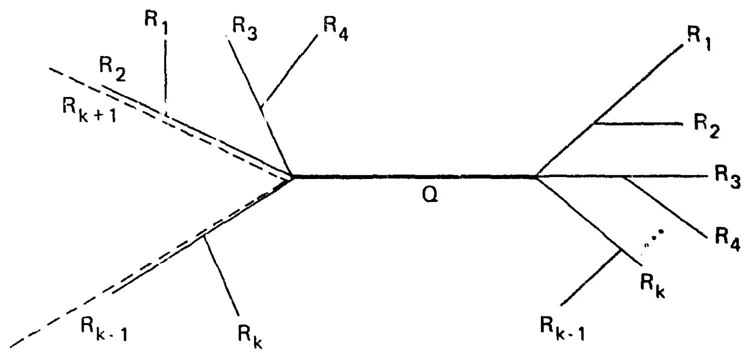

Fig. 2.

If R_{k+1} has no non-trivial intersection with Q, then always three intervals can be found which do not have a non-trivial intersection (see Fig. 2) what contradicts the assumption. □

The observation has been applied to a problem concerning the intersection graphs of fundamental cycles of a graph over the set of edges of the graph, for details see [3].

Gyarfas and Lehel [2] informed the author about a new parameter introduced for a family of subtrees of a tree which generalizes the result included in the observation.

References

[1] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).

[2] A. Gyárfás and J. Lehel, Private communication, 1979.

[3] M.M. Sysło, On some problems related to fundamental cycle sets of a graph, in: G. Chartrand, Y. Alavi, D. Goldsmith, L. Lesniak-Foster and D.R. Lick, eds., The Theory and Applications of Graphs (J. Wiley, New York, 1981).